$$x(t) = X e^{s_1 t} \rightarrow y_{ss}(t) = X H(s_1) e^{s_1 t} \quad ; t \geq 0$$

1(a)/ $x(t) = 5 = 5 e^{s_1 t} = 5 e^{(0)t}$

(i) $y_{ss}(t) = 5 \cdot \frac{4}{(0)+5} e^{(0)t} = 4 \quad ; t \geq 0$

(ii) $y_{ss}(t) = 5 \cdot \left. \frac{s+5}{s^2+2s+5} \right|_{s=0} \cdot e^{(0)t} = 5 \quad ; t \geq 0$

(b)/ $x(t) = e^{-3t} u(t) = 1 \cdot e^{s_1 t} = 1 \cdot e^{(-3)t} u(t)$

(i) $y_{ss}(t) = 1 \cdot \frac{4}{-3+5} \cdot e^{-3t} u(t) = 2 e^{-3t} u(t)$

(ii) $y_{ss}(t) = 1 \cdot \frac{-3+5}{(-3)^2+2\cdot(-3)+5} \cdot e^{-3t} u(t) = \frac{2}{9-6+5} e^{-3t} u(t)$

$$= \frac{1}{4} e^{-3t} u(t).$$

2/ $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$\hat{x}(t) = x(t-t_0)$

$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \hat{c}_k e^{jk\omega_0 t}$ ——— (I)

now, $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$\Rightarrow x(t-t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 (t-t_0)}$

$\Rightarrow x(t-t_0) = \sum_{k=-\infty}^{\infty} \left(c_k e^{-jk\omega_0 t_0}\right) e^{jk\omega_0 t}$

$\Rightarrow \hat{x}(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left(c_k e^{-jk\omega_0 t_0}\right)}_{\hat{c}_k} e^{jk\omega_0 t}$ ——— (II) comparing (i) and (ii) $\Rightarrow$

$$\hat{c}_k = c_k e^{-jk\omega_0 t_0} \quad \text{—— (iii)}$$

$$\Rightarrow |\hat{c}_k| = |c_k e^{-jk\omega_0 t_0}| = |c_k| \, |e^{-jk\omega_0 t_0}|$$

$$= |c_k| \, |\cos(k\omega_0 t_0) - j\sin(k\omega_0 t_0)|$$

$$= |c_k| \sqrt{\cos^2(k\omega_0 t_0) + \sin^2(k\omega_0 t_0)}$$

$$\Rightarrow \boxed{|\hat{c}_k| = |c_k|}$$

now, $c_k = |c_k| e^{\overbrace{j\theta_k}^{\text{phase}}}$ —— (iv) $= |c_k| \angle \theta_k$ ← phase

and $\hat{c}_k = |\hat{c}_k| e^{j\hat{\theta}_k} = |\hat{c}_k| \angle \hat{\theta}_k = c_k e^{-jk\omega_0 t_0}$ [from (iii)]

$$\Rightarrow \hat{c}_k = c_k e^{-jk\omega_0 t_0} = |c_k| e^{j\theta_k} e^{-jk\omega_0 t_0} \quad \text{(from (iv))}$$

$$\Rightarrow \hat{c}_k = |c_k| e^{j(\theta_k - k\omega_0 t_0)} = |c_k| \angle(\theta_k - k\omega_0 t_0)$$

$$\Rightarrow \hat{c}_k = |c_k| \angle(\theta_k - k\omega_0 t_0)$$

$$\Rightarrow \boxed{\angle \hat{c}_k = \angle c_k - k\omega_0 t_0} \quad ; \angle c_k = \theta_k$$

3/ $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} c_{-k} e^{jk\omega_0 t} \quad \text{—— (i)}$$

now, $$x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos k\omega_0 t + B_k \sin k\omega_0 t]$$

$$= A_0 + \sum_{k=1}^{\infty} \left[A_k \frac{e^{jk\omega_0 t} + e^{-jk\omega_0 t}}{2} + B_k \frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j} \right]$$

$$= A_0 + \sum_{k=1}^{\infty} e^{jk\omega_0 t} \left(\frac{A_k}{2} - j\frac{B_k}{2} \right) + \sum_{k=-\infty}^{-1} e^{jk\omega_0 t} \left(\frac{A_k}{2} + j\frac{B_k}{2} \right)$$

—— (ii)

comparing ① & ⑪ ⇒

$C_0 = A_0$

$$\left.\begin{aligned} C_k &= \frac{A_k}{2} - j\frac{B_k}{2} \\ C_{-k} &= \frac{A_k}{2} + j\frac{B_k}{2} \end{aligned}\right\} \Rightarrow C_k = C_{-k}^* = \frac{A_k}{2} - j\frac{B_k}{2}$$

and $C_0 = A_0$.

4/ (a) $\int_0^{2\pi = T} \sin^2(t)\,dt = \int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\,dt$

$= \frac{1}{2}\int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \cos 2t\,dt$

$= \frac{1}{2}[t]_0^{2\pi} - \frac{1}{2}\cdot\frac{1}{2}[\sin 2t]_0^{2\pi}$

$= \frac{1}{2}\cdot 2\pi - \frac{1}{4}[\sin 4\pi - \sin 0] = \frac{1}{2}\cdot 2\pi = \frac{1}{2}T$

(b) $\int_0^{2\pi = T} \sin^2(2t)\,dt = \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\cdot 2t)\,dt$

$= \frac{1}{2}\int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \cos 4t\,dt$

$= \frac{1}{2}\cdot 2\pi - \frac{1}{2}\cdot\frac{1}{4}[\sin 4t]_0^{2\pi} = \frac{1}{2}\cdot 2\pi - [\sin 8\pi - \sin 0]$

$= \frac{1}{2}\cdot 2\pi = \frac{1}{2}\cdot T$

(c) $I = \int \sin t \sin 2t\,dt$ ⟶ ①

we know $\int u\,dv = uv - \int v\,du$

let's assume, $u = \sin 2t \Rightarrow du = 2\cos 2t\,dt$

and $dv = \sin t\,dt$

$\Rightarrow v = -\cos t$

so, $I = \int \sin t \sin 2t\,dt = \int dv\,u$

$= \sin 2t\cdot(-\cos t) - \int (-\cos t)(2\cos 2t)\,dt$

$\Rightarrow I = -\sin 2t\cdot\cos t + 2\int \cos t\cdot\cos 2t\,dt$ ⟶ ⑪

again, let's assume,

$$u_1 = \cos 2t \Rightarrow du_1 = -2\sin 2t\, dt$$

and $dv_1 = \cos t\, dt$

$$\Rightarrow v_1 = \sin t$$

so from (11) $\Rightarrow$

$$I = -\sin 2t \cdot \cos t + 2\left\{ \cos 2t \cdot \sin t - \int \sin t\,(-2\sin 2t)\,dt \right.$$

$$= -\sin 2t \cdot \cos t + 2\cos 2t \cdot \sin t + 4\underbrace{\int \sin t \cdot \sin 2t \cdot dt}_{I}$$

$$\Rightarrow I = -\sin 2t \cdot \cos t + 2\cos 2t \cdot \sin t + 4I$$

$$\Rightarrow I = \frac{-1}{3}\left[-\sin 2t \cdot \cos t + 2\cos 2t \cdot \sin t\right]$$

$$\Rightarrow \int \sin t \sin 2t\, dt = \frac{-1}{3}\left[-\sin 2t \cdot \cos t + 2\cos 2t \cdot \sin t\right]$$

$$\Rightarrow \int_0^{2\pi = T} \sin t \cdot \sin 2t\, dt = \left[-\frac{2}{3}\cos 2t \cdot \sin t + \frac{1}{3}\sin 2t \cdot \cos t\right]_0^{2\pi}$$

$$\Rightarrow \int_0^{2\pi} \sin t \cdot \sin 2t\, dt = \frac{-2}{3} \cdot \cos 4\pi \cdot \sin 2\pi + \frac{1}{3}\sin 4\pi \cos 2\pi$$
$$+ \frac{2}{3} \cdot \cos 0 \cdot \sin 0 - \frac{1}{3}\sin 0 \cdot \cos 0$$

$$\Rightarrow \int_0^{2\pi} \sin t \cdot \sin 2t\, dt = 0$$

(d) let's assume,

$$x_1(t) = \sin k_1 t$$

and $x_2(t) = \sin k_2 t$

a period of x_1 and x_2 expands from 0 to 2π

now, $\int_0^{2\pi=T} x_1(t)\cdot x_2(t)\,dt = \int_0^{2\pi=T} \sin k_1 t \cdot \sin k_2 t\,dt$

$$= \begin{cases} 0, & \text{when } k_1 \neq k_2 \quad (4c) \\ \frac{T}{2} = \pi, & \text{when } k_1 = k_2 \quad (4a, 4b) \end{cases}$$

this shows $x_i(t) = \sin k_i t$ has orthogonality property.

From Euler's formula.

$$\sin k_i t = \frac{1}{2j}\left(e^{jk_i t} - e^{-jk_i t}\right)$$

as $\sin k_i t$ has orthogonality property, so does its components $e^{jk_i t}$ and $e^{-jk_i t}$.

$$\int_{T_0} e^{j(n-m)\omega_0 t}\,dt = \int_{T_0} \cos(n-m)\,\omega_0 t\,dt + j\int_{T_0} \sin(n-m)\omega_0 t\,dt$$

if $n = m$ $\quad \int_{T_0} e^{j(n-m)\omega_0 t}\,dt = \int_{T_0} \cancelto{1}{\cos(0)}\,dt + j\int_{T_0} \cancelto{0}{\sin(0)}\,dt$

$$= T_0$$

if $n \neq m$, $\quad \int_{T_0} e^{j(n-m)\omega_0 t}\,dt = \int_{T_0} \cos(n-m)\omega_0 t\,dt + j\int \sin(n-m)\omega_0 t\,dt$

$$= 0 + 0 = 0$$

$$\left[\because \int_{T_0} \cos \omega t = \int_{T_0} \sin \omega t = 0\right]$$

$$\Rightarrow \int_{T_0} e^{j(n-m)\omega_0 t}\,dt = \begin{cases} 0, & m \neq n \\ T_0; & n = m \end{cases}$$